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Abstract

This paper is concerned primarily with determining the mutual coherence function of the field produced by a plane quasi-monochromatic source in a region of variable refractive index. A scalar theory is used throughout.

A brief review of the conceptual background of coherence theory is presented in Section 1. Section 2 contains an outline of the mathematical formalism of coherence theory and shows that in a region of variable refractive index the mutual coherence function is propagated according to a pair of inhomogeneous scalar wave equations. In Section 3 the pair of wave equations are solved with appropriate Green's functions. An expression is derived for the mutual coherence function of a field produced by a plane quasi-monochromatic source. In Section 4 the case of a statistically homogeneous medium is treated and an expression for the ensemble average of the mutual coherence function is obtained in terms of integrals of the two-point correlation function characterizing the medium.

ON THE MUTUAL COHERENCE FUNCTION IN AN INHOMOGENEOUS MEDIUM

1. Introduction

The development of coherence theory has been strongly influenced by research in visual optics. The theory is concerned with the behavior of electromagnetic fields at frequencies so high that measurements can be made only by averaging intensities over periods of time that are long compared with the times involved for individual fluctuations of the fields. The fields are assumed to have stationary time dependence, at least for the intervals of the averaging periods.

In an advanced theory of optical behavior, the concepts of amplitude and phase, which are often helpful in other branches of electromagnetic theory and in elementary optics, are no longer useful. The usefulness of these concepts breaks down not only because the high frequencies involved in optics make it impossible to measure the amplitude and phase of field components, but even more fundamentally, light in its usual form (the superposition of a large number of randomly timed statistically independent pulses) is not strictly monochromatic but consists of spectra of finite widths. Indeed, under the conditions stated it cannot be analyzed even by a Fourier decomposition into strictly monochromatic components because only the power carried by a narrow band of wavelengths can be measured.

In the more elementary theory of optical behavior, the concepts of amplitude and phase presuppose a strictly monochromatic source of illumination and are not measurable quantities (at least in the optics realm). The basic quantities in coherence theory (time-averaged intensities and functions that express the degree of correlation between the vibrations at different points in the field), however, are measurable and do not presuppose the nature of the field. Furthermore, as limiting cases, coherence theory yields not only a strictly monochromatic theory, but also a theory of incoherent radiation (addition of intensities) and gives an accurate description of the region of partially coherent light between these two limits.

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2. Outline of Mathematical Formalism of the Theory

The fundamental entities of coherence theory — the mutal coherence function and the complex degree of coherence — will be introduced in this section, and the wave equations for the propagation of the mutual coherence function in an inhomogeneous medium will be derived. For the most part, theorems will be stated and the reader referred to the literature for proofs and further discussion.

In a source-free, although not necessarily homogeneous, medium the real scalar function of position and time $V^r(\underline{P}, t)$ satisfies the scalar wave equation

$$\nabla^2 V^r(\underline{P}, t) = \frac{1}{C^2(\underline{P})} \frac{\partial^2 V^r(\underline{P}, t)}{\partial t^2}$$

where $\underline{P}$ is the position vector. (It will occasionally be convenient to indicate position by a subscript or to omit explicit spatial dependence.)

The intensity of $V^r(t)$ averaged over an interval of time of length 2T is given by the expression

$$\frac{1}{2T} \int_{-T}^{T} [V^r(\underline{P}, t)]^2 \, dt.$$

In the following applications, T will be extremely large in terms of time units of the order of the actual fluctuations (for example, the mean period $1/\bar{\nu}$ where $\bar{\nu}$ is the mean frequency of the disturbance). It is therefore convenient to let $T \to \infty$ in expressions for the time-averaged intensity and for the correlation functions to be introduced below. It is assumed, of course, that the

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [V^r(\underline{P}, t)]^2 dt$$

is finite.

A complex function $V(\underline{P}, t)$, the analytic signal, is associated with the real function $V^r(\underline{P}, t)$. The advantages of choosing the analytic signal as a complex representation of the disturbance have already been discussed at length.[2] The analytic signal may be defined in terms of Hilbert transforms as follows:

Let $f^r(t)$ be a real function of time such that its Hilbert transform exists (a sufficient condition is square integrability); the analytic signal, $f(t)$ [associated with $f^r(t)$], is defined to be $f(t) = f^r(t) + if^i(t)$, where $f^i(t)$ is the Hilbert transform of $f^r(t)$; that is,

$$f^i(t) = \frac{-1}{\pi} \int_{-\infty}^{+\infty} \frac{f^r(t')}{t'-t} \, dt' \equiv H[f^r(t)].$$

The integral sign $\int^*$ denotes that the Cauchy principal value is taken at $t'=t$.

Although $V^r(t)$ is in general not square integrable, it is possible to employ an analytic signal representation by the stratagem of truncating $V^r(t)$ at a particular value, $t=T$, and then let $T \to \infty$. Thus, let

$$V^r(\underline{P};\, T,\, t) = \begin{cases} V^r(\underline{P},\, t) & |t| \leq T \\ 0 & |t| > T \end{cases}$$

and the associated analytic signal $V(P;T,t)$ be given by

$$V(\underline{P};\, T,\, t) = V^r(\underline{P};\, T,\, t) + i\,{}_T V^i(\underline{P},\, t)$$

where

$${}_T V^i(\underline{P},\, t) = H[V^r(\underline{P};\, T,\, t)].$$

It should be noted that ${}_T V^i(\underline{P},\, t)$ is not a truncated function.

In terms of these functions the basic quantities of coherence theory, the mutual coherence function, and the complex degree of coherence may be precisely defined. The mutual coherence function $\Gamma(\underline{P}_1,\, \underline{P}_2,\, \tau) \equiv \Gamma_{12}(\tau)$ is defined as the complex cross correlation between the analytic signal representation of the real field at the two points $\underline{P}_1$ and $\underline{P}_2$; that is,

$$\Gamma_{12}(\tau) = \lim_{T \to \infty} \int_{-\infty}^{\infty} V_1(T, t+\tau)\; V_2^*(T,\, t)\; dt.$$

The time average and limiting process is denoted by sharp brackets:

$$\Gamma_{12}(\tau) = \langle V_1(t+\tau)\; V_2^*(t) \rangle .$$

By use of the theorem that the cross correlation of the two real functions is equal to the cross correlation of their Hilbert transforms in the same order,[3] it can be readily shown that the time-averaged intensity I_m at the point P_m is given by $1/2\; \Gamma_{mm}(0)$; that is,

$$I_m = \langle [V_m{}^r(t)]^2 \rangle = 1/2\;\; \Gamma_{mm}(0),\;\; (m = 1\;\; 2)\;.$$

Also, by use of the theorem that the convolution of two analytic signals is itself an analytic signal, it can be shown[3] that $\Gamma_{12}(\tau)$ is an analytic signal. From the last result it follows[1] that $\Gamma_{12}(\tau)$ possesses a Fourier spectrum that is zero for half the frequency range. Thus

$$\Gamma_{12}(\tau) = \int_{-\infty}^{\infty} \hat{\Gamma}_{12}(\nu) e^{-2\pi i \nu \tau}\, d\nu$$

where

$$\hat{\Gamma}_{12}(\nu) = \int_{-\infty}^{\infty} \Gamma_{12}(\tau)\, e^{2\pi i\nu\tau} d\tau \equiv 0 \;\; \nu < 0.$$

It should be noted here that in formulating the solution to actual problems, expressions of the form $\langle V_1(t_1+t)\; V_2{}^*(t_2+t)\rangle$ are usually obtained. Under the change of variables $t' = t + t_2$, $\langle V_1(t^1+\tau)\; V_2{}^*(t^1)\rangle$ is obtained, where $\tau = t_2 - t_1$. The additional assumption of stationarity of V(t) (that is, the time averages are independent of the choice of time origin, or equivalently that the time averages are a function of the difference in time only) is necessary to equate $\Gamma_{12}(\tau)$ with $\langle V_1(t_1+t)\; V_2{}^*(t_2+t)\rangle$.

A normalized form of the mutual coherence function, called the complex degree of coherence and denoted by $\gamma_{12}(\tau)$, is very useful in coherence theory; $\gamma_{12}(\tau)$ is defined to be

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\;\;\Gamma_{22}(0)}}\,.$$

With the Schwartz inequality it can be shown that $0 \leq |\gamma_{12}(\tau)| \leq 1$. The limits characterize incoherent radiation and coherent radiation respectively.

The important result that $\Gamma_{12}(\tau)$ is propagated according to a pair of wave equations will now be proved. Specifically:

$$\nabla_m{}^2 \Gamma(\underline{P}_1, \underline{P}_2, \tau) = \frac{1}{C^2(\underline{P}_m)} \; \frac{\partial^2 \Gamma(\underline{P}_1, \underline{P}_2, \tau)}{\partial \tau^2}, \quad (m = 1, 2). \qquad (2.1)$$

Here the Laplacian $\nabla_m{}^2$ acts on the coordinates of the point $\underline{P}_m$ (m = 1, 2), and the spatial dependence of the velocity of propagation is indicated.

To prove Eq. (2.1), it is assumed that $V^r(\underline{P}, t)$, hence the truncated function $V^r(\underline{P}; T, t)$ satisfies the scalar wave equation

$$\nabla^2 V^r(\underline{P}; T, t) = \frac{1}{C^2(\underline{P})} \; \frac{\partial^2 V^r(\underline{P}; T, t)}{\partial t^2}. \qquad (2.2)$$

Also, as defined above

$${}_T V^i(\underline{P}, t) = \frac{1}{\pi} \int_{-\infty}^{*\infty} \frac{V^r(\underline{P}; T, t')}{t' - t} dt'. \qquad (2.3)$$

Both sides of Eq. (2.3) are operated on with the Laplacian ∇^2, the order of operations is interchanged, and with the theorem[4] that the Hilbert transform of the derivative of a function equals the derivative of the Hilbert transform of the function, the following is obtained:

$$\nabla_T^2 V^i(\underline{P}, t) = \frac{1}{C^2(\underline{P})} \frac{\partial_T{}^2 V^i(\underline{P}, t)}{\partial t^2}. \tag{2.4}$$

Multiplication of Eq. (2.4) by i and addition to Eq. (2.2) yields

$$\nabla^2 V(\underline{P}; T, t) = \frac{1}{C^2(\underline{P})} \frac{\partial^2 V(\underline{P}; T, t)}{\partial t^2}. \tag{2.5}$$

Thus, the analytic signal itself satisfies the wave equation.

Now

$$\Gamma(\underline{P}_1, \underline{P}_2, \tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} V(\underline{P}_1; T, t+\tau)\, V^*(\underline{P}_2; T, t)\, dt. \tag{2.6}$$

Equation (2.6) is differentiated with respect to $\underline{P}_1$, the order of operations is interchanged, and Eq. (2.5) is substituted, yielding

$$\begin{aligned}
\nabla_1^2 \Gamma_{12}(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \nabla_1^2 [V_1(T, t+\tau)]\, V_2^*(T, t)\, dt \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \frac{1}{C^2(\underline{P}_1)} \frac{\partial^2 V_1(T, t+\tau)}{\partial \tau^2} V_2^*(T,t) dt \\
&= \frac{1}{C^2(\underline{P}_1)} \frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau).
\end{aligned}$$

Similarly,

$$\nabla_2^2 \Gamma_{12}(\tau) = \frac{1}{C^2(\underline{P}_2)} \frac{\partial^2 \Gamma_{12}(\tau)}{\partial \tau^2},$$

and the proof is complete.

3. Mutual Coherence Function in an Inhomogeneous Medium

The central problem of this paper is the determination of the mutual coherence function for a field produced by an extended polychromatic source in a region of variable index of refraction. In the discussion, S is an arbitrary surface containing an extended polychromatic source with a known distribution of mutual coherence; $\underline{P}_1$ and $\underline{P}_2$ are points in the illuminated field V; and $\underline{S}_1$ and $\underline{S}_2$ are points on the surface S.

As shown above, the propagation of the mutual coherence function in a source-free but inhomogeneous medium is governed by the pair of wave equations

$$\nabla_m^2 \Gamma_{12}(\tau) = \frac{1}{C^2(\underline{P}_m)} \frac{\partial^2 \Gamma_{12}(\tau)}{\partial \tau^2}, \quad m = 1, 2. \tag{3.1}$$

It is assumed that $\Gamma_{12}(\tau)$ is known for all pairs of points $\underline{S}_1$ and $\underline{S}_2$ on the surface S.

Let $\hat{\Gamma}_{12}(\nu)$ be the Fourier transform of $\Gamma_{12}(\tau)$. Then, as stated above, since $\Gamma_{12}(\tau)$ is an analytic signal, its Fourier spectrum contains positive frequencies only; that is,

$$\Gamma_{12}(\tau) = \int_0^\infty \hat{\Gamma}_{12}(\nu)\, e^{-2\pi i\nu\tau} d\nu \tag{3.2}$$

where

$$\hat{\Gamma}_{12}(\nu) = \int_{-\infty}^\infty \Gamma_{12}(\tau) e^{2\pi i\nu\tau} d\tau. \tag{3.3}$$

Substitution from Eq. (3.2) and Eq. (3.1) and interchange of the order of integration and differentiation yields

$$\int_0^\infty [\nabla_m{}^2 + k_m{}^2(\nu)]\; \hat{\Gamma}_{12}(\nu) e^{-2\pi i\nu\tau}\, d\nu = 0 \;(m = 1, 2). \tag{3.4}$$

Since Eq. (3.4) holds for all τ,

$$[\nabla_1{}^2 + k^2(\underline{P}_1, \nu)]\; \hat{\Gamma}(\underline{P}_1, \underline{P}_2, \nu) = 0 \tag{3.5a}$$

$$[\nabla_2{}^2 + k^2(\underline{P}_2, \nu)]\; \hat{\Gamma}(\underline{P}_1, \underline{P}_2, \nu) = 0. \tag{3.5b}$$

Here $k(\underline{P}, \nu) = 2\pi\nu/C(\underline{P})$. Thus, each spectral component of $\Gamma_{12}(\tau)$ satisfies the pair of Helmholtz equations, Eqs. (3.5a and b).

In Eq. (3.5a), $\underline{P}_2$ is a fixed parameter so far as the operator is concerned. In particular, Eq. (3.5a) holds if $\underline{P}_2$ is a fixed point $\underline{S}_2$ on a closed surface S. The equation then becomes

$$[\nabla_1{}^2 + k^2(\underline{P}_1, \nu)]\; \hat{\Gamma}(\underline{P}_1, \underline{S}_2, \nu) = 0. \tag{3.6}$$

The boundary condition for Eq. (3.6) is the known values of $\hat{\Gamma}(\underline{S}_1, \underline{S}_2, \nu)$. Hence, the problem is to solve the pair of equations

$$[\nabla_1{}^2 + k^2(\underline{P}_1, \nu)]\, \hat{\Gamma}(\underline{P}_1, \underline{S}_2, \nu) = 0 \tag{3.7a}$$

with $\hat{\Gamma}(\underline{S}_1, \underline{S}_2, \nu)$ known on the boundary, and

$$[\nabla_2{}^2 + k^2(\underline{P}_2, \nu)]\; \hat{\Gamma}(\underline{P}_1, \underline{P}_2, \nu) = 0 \tag{3.7b}$$

with $\hat{\Gamma}(\underline{P}_1, \underline{S}_2, \nu)$ known on the boundary as a result of solving Eq. (3. 7a).

A formal solution to Eqs. (3. 7a and b) can be obtained easily in terms of the Green's function $G(\underline{P}, \underline{P}')$ which satisfies the equation

$$[\nabla^2 + k^2(\underline{P})]\, G(\underline{P}, \underline{P}') = -\delta(\underline{P}-\underline{P}') \tag{3.8}$$

and which vanishes on the boundary S. In the case that S is plane, the Green's function must be chosen not only to vanish on the boundary plane but also to satisfy the radiation condition at infinity.

In the same way as if k were a constant, the following is obtained:

$$\hat{\Gamma}(\underline{P}_1, \underline{P}_2) = \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1' \underline{S}_2') \frac{\partial G_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'} \frac{\partial G_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} dS_1'\, dS_2'. \tag{3.9}$$

In Eq. (3. 9) and in following equations primes are used to indicate the variables of integration, and corresponding subscripts and primes attached to the surface $\underline{S}$ also serve this purpose. Explicit dependence on the frequency ν has been omitted for conciseness. It will be shown below that $G_2 = G_1{}^*$.

An explicit form for $G(\underline{P}, \underline{P}')$ can be obtained using an iterative procedure. Equation (3. 8) is rewritten to represent $k^2(\underline{P}, \nu)$ as the sum of a fixed mean value $k^2(\nu)$ and a variable part with zero mean, $k^2(\nu)\epsilon(\underline{P}, \nu)$. Thus,

$$(\nabla^2 + k^2)\, G(\underline{P}, \underline{P}') = -\delta(\underline{P}-\underline{P}') - k^2 \epsilon(\underline{P})\, G(\underline{P}, \underline{P}'). \tag{3.10}$$

(Physically $\epsilon(\underline{P})$ can be said to be the variable part of the dielectric constant of the medium.) Now Eq. (3. 10) can be formally taken to be an inhomogeneous constant-coefficient Helmholtz equation with the right-hand side as the source term. Acc ~dingly, a solution to Eq. (3. 10) can be obtained in terms of the Green's fu.. n $g(P, P'')$ which satisfies the constant coefficient equation

$$(\nabla^2 + k^2)\, g(\underline{P}, \underline{P}'') = -\delta(\underline{P}-P'') \tag{3.11}$$

and vanishes on the boundary surface. Since g also vanishes on the surface,

$$G(\underline{P}, \underline{P}') = g(\underline{P}, \underline{P}') + k^2 \int_{V''} \epsilon(\underline{P}'')\, G(\underline{P}'', \underline{P}')\, g(\underline{P}, \underline{P}'')\, dV''. \tag{3.12}$$

Equation (3. 12) can now be used as the basis for an iterative development:

$$\begin{aligned} G(\underline{P}, \underline{P}') = {} & g(\underline{P}, \underline{P}') + k^2 \int_{V''} \epsilon(\underline{P}'')\, g(\underline{P}'', \underline{P}')\, g(\underline{P}'', \underline{P})\, dV'' \\ & + k^4 \int_{V''} \int_{V'''} \epsilon(\underline{P}'')\, \epsilon(\underline{P}''')\, g(\underline{P}''', P'')\, g(\underline{P}'', \underline{P}')\, g(\underline{P}'', \underline{P})\, dV'''\, dV'' \\ & + \ldots \end{aligned} \tag{3.13}$$

Substitution of the iterative series Eq. (3. 13) into Eq. (3. 9) yields the following expression for the Fourier transform of the mutual coherence function:

$$
\begin{aligned}
\hat{\Gamma}(\underline{P}_1, \underline{P}_2) = & \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \frac{\partial g_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'} \frac{\partial g_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} \, d\underline{S}_1' \, d\underline{S}_2' \\
& + k^2 \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \frac{\partial g_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} \int_{V_1''} \epsilon(\underline{P}_1'') \, g_1(\underline{P}_1'', \underline{P}_1) \frac{\partial g_1(\underline{P}_1'', \underline{S}_1')}{\partial \eta_1'} \, dV_1'' \, d\underline{S}_1' \, d\underline{S}_2' \\
& + k^2 \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \frac{\partial g_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'} \int_{V_2''} \epsilon(\underline{P}_2'') \, g_2(\underline{P}_2'', \underline{P}_2) \frac{\partial g_2(\underline{P}_2'', \underline{S}_2')}{\partial \eta_2'} \, d\underline{V}_2'' \, d\underline{S}_1' \, d\underline{S}_2' \\
& + k^4 \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \frac{\partial g_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} \int_{V_1''} \int_{V_1'''} \epsilon(\underline{P}_1'') \, \epsilon(\underline{P}_1''') \, g_1(\underline{P}_1''', \underline{P}_1'') \, g_1(\underline{P}_1'', \underline{P}_1) \frac{\partial g_1}{\partial \eta_1'} \, d\underline{V}_1''' \, d\underline{V}_1'' \, d\underline{S}_1' \, d\underline{S}_2' \\
& + k^4 \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \int_{V_1''} \int_{V_2''} \epsilon(\underline{P}_1'') \, \epsilon(\underline{P}_2'') \, g_1(\underline{P}_1'', \underline{P}_1) \, g_2(\underline{P}_2'', \underline{P}_2) \frac{\partial g_1(\underline{P}_1'', \underline{S}_1')}{\partial \eta_1'} \frac{\partial g_2(\underline{P}_2'', \underline{S}_2')}{\partial \eta_2'} \, d\underline{V}_2'' \, d\underline{V}_1'' \, d\underline{S}_2' \, d\underline{S}_1' \\
& + k^4 \int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \frac{\partial g_1(P_1, S_1')}{\partial \eta_1} \int_{V_2''} \int_{V_2'''} \epsilon(\underline{P}_2'') \, \epsilon(\underline{P}_2''') \, g_2(\underline{P}_2''', \underline{P}_2'') \, g_2(\underline{P}_2'', \underline{P}_2) \frac{\partial g_2}{\partial \eta_2'} \, d\underline{V}_2''' \, d\underline{V}_2'' \, d\underline{S}_1' \, d\underline{S}_2' \\
& + \ldots
\end{aligned}
\tag{3.14}
$$

The Green's function $g(\underline{P}, \underline{P}')$ has been determined[2] for the important case in which S is a plane surface. The result is given here:

$$
g(\underline{P}, \underline{P}') = \frac{e^{\pm ik|\underline{P}-\underline{P}'|}}{|\underline{P}-\underline{P}'|} - \frac{e^{\pm ik|\underline{P}_i-\underline{P}'|}}{|\underline{P}_i-\underline{P}'|} \tag{3.15}
$$

The image of the point $\underline{P}$ in the plane S is denoted by $\underline{P}_i$ (see Fig. 1). In Eq. (3.14) the plus sign is taken where a subscript 1 appears, and the minus sign is taken where a subscript 2 appears.

It should be noted that an identical expression for $\hat{\Gamma}(\underline{P}_1, \underline{P}_2, \nu)$ can be obtained by a procedure that iterates for the transform of the mutual coherence function itself rather than for the Green's function of the Helmholtz equation with a variable coefficient. This procedure puts Eqs. (3.7a and b), the pair of differential equations, in the form of a pair of equivalent integral equations by use of Green's functions. Thus,

$$\hat{\Gamma}(\underline{P}_1, \underline{S}_2, \nu) = -\int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2, \nu) \frac{\partial g_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'} dS_1'$$

$$+ k^2 \int_{V_1'} \epsilon(\underline{P}_1') \hat{\Gamma}(\underline{P}_1', \underline{S}_2)\, g_1(\underline{P}_1, \underline{P}_1')\, dV_1', \tag{3.16a}$$

$$\hat{\Gamma}(\underline{P}_1, \underline{P}_2, \nu) = -\int_{\underline{S}_2'} \hat{\Gamma}(\underline{P}_1, \underline{S}_2') \frac{\partial g_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} d\underline{S}_2'$$

$$+ k^2 \int_{V_2'} \epsilon(\underline{P}_2') \hat{\Gamma}(\underline{P}_1, \underline{P}_2')\, g_2(\underline{P}_2, \underline{P}_2')\, dV_2'. \tag{3.16b}$$

Here, the Green's function $g(\underline{P}, \underline{P}')$ is identical to the Green's function of Eq. (3.11), and the representation $k^2(\underline{P}) = k^2[1 + \epsilon(\underline{P})]$ has again been employed. The integral equations, Eqs. (3.16a and b), are then used as the basis for an interative procedure that starts with the surface terms as a zeroth order approximation and by successive substitutions yields the same iterative series for $\hat{\Gamma}(\underline{P}_1, \underline{P}_2)$ as obtained above.

The form of the iterative solution, Eq. (3.14), obtained for the propagation of the mutual coherence function in a medium in which the refractive index varies is that of the uniform space solution

$$\int_{\underline{S}_2'} \int_{\underline{S}_1'} \hat{\Gamma}(\underline{S}_1', \underline{S}_2') \frac{\partial g_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'} \frac{\partial g_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} d\underline{S}_1'\, d\underline{S}_2'$$

modified by a series of correction terms that involve volume integrals of a quantity associated with the fluctuations of the refractive index. When the refractive index is constant, the correction terms become zero and the solution reduces to the uniform space solution.

The iterative solution obtained is that for a single spectral component of $\Gamma_{12}(\tau)$. To obtain $\Gamma_{12}(\tau)$, the iterative solution $\hat{\Gamma}_{12}(\tau)$ must be substituted in Eq. (3.2) and the integration over ν performed. However, in the case of quasi-monochromatic fields, it is unnecessary to perform the actual integration. A quasi-monochromatic field is one for which the effective spectral range, $\Delta\nu$, is small compared with the mean frequency, $\bar{\nu}$; that is, $\Delta\nu/\bar{\nu} << 1$. For this case, provided that the time difference τ is small compared with the coherence time $1/\Delta\nu$, it is known [1, 2] that the mutual coherence function is of the form

$$\Gamma_{12}(\tau) \simeq \Gamma_{12}(0)e^{-2\pi i\bar{\nu}\tau}, |\tau| << 1/\Delta\nu \tag{3.17}$$

where $\bar{\nu}$ is the mean frequency of $\Gamma_{12}(\tau)$. Substituting from Eq. (3.17) into Eq. (3.1) yields

$$[\nabla_m^{\ 2} + k^2(\underline{P}_m, \bar{\nu})]\, \Gamma_{12}(0) = 0 \ (m = 1, 2) \tag{3.18}$$

where

$$k\,(\underline{P}_m, \bar{\nu}) = \frac{2\pi\bar{\nu}}{C\,(\underline{P}_m)}\,.$$

Thus, under the quasi-monochromatic approximation of narrow spectral width and small path differences, it has been shown that $\Gamma_{12}(0)$ satisfies the same pair of Helmholtz equations, Eq. (3.5), as does $\hat{\Gamma}_{12}(\nu)$ with ν now fixed at the mean frequency $\bar{\nu}$. The boundary condition becomes $\Gamma(\underline{S}_1, \underline{S}_2, 0)$ and the iterative solution yields $\Gamma_{12}(0)$ which when substituted in Eq. (3.17) yields the quasi-monochromatic solution. (It should be noted that the varying propagation velocity implies that the time difference τ may be different for two paths of the same geometric length.)

The Green's function formulation of the solution in Eq. (3.9) leads directly to an important result which we state as a theorem.

Theorem

The field produced by a quasi-monochromatic coherent source extended over a surface in contact with a source-free time-invariant medium with arbitrary refractive index variation (in space) is itself coherent.

Proof

The proof of this theorem starts with the result[2] that a quasi-monochromatic field (source) is coherent if, and only if, the mutual intensity $\Gamma_{12}(0)$ can be represented as the product of a wave function U, evaluated at $\underline{P}_1$, with its complex conjugate U*, evaluated at $\underline{P}_2$; that is

$$\Gamma(\underline{P}_1, \underline{P}_2, 0) = U(\underline{P}_1)\ U^*(\underline{P}_2).$$

In the quasi-monochromatic approximation it is seen from Eq. (3.9) and the discussion above that

$$\Gamma(\underline{P}_1, \underline{P}_2, \tau) \simeq e^{-2\pi i\bar{\nu}\tau} \int_{\underline{S}_2'} \int_{\underline{S}_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0)\, \frac{\partial G_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'}\, \frac{\partial G_2(\underline{P}_2, \underline{S}_2')}{\partial \eta_2'} \times d\underline{S}_1'\, d\underline{S}_2' \left(|\tau| << \frac{1}{\Delta\nu}\right). \tag{3.19}$$

The assumption that the source is coherent allows the factorization

$$\Gamma(\underline{P}_1, \underline{P}_2, 0) = \left[\int_{S'} U(\underline{S'}) \frac{\partial G_1(\underline{P}_1, \underline{S'})}{\partial \eta'} dS'\right]\left[\int_{S'} U^*(\underline{S'}) \frac{\partial G_2(\underline{P}_2, \underline{S'})}{\partial \eta'} dS'\right] \quad (3.20)$$

where the notation has been changed slightly to make the relationship between the bracketed quantities more apparent. The proof will be complete if it can be shown that

$$\frac{\partial G_2}{\partial \eta'} = \frac{\partial G_1^*}{\partial \eta'} .$$

To prove this relationship the following lemma[3] is used:

$$\Gamma^*(\underline{P}_1, \underline{P}_2, 0) = \Gamma(\underline{P}_2, \underline{P}_1, 0). \quad (3.21)$$

Substitution from Eq. (3.20) into Eq. (3.21), yields

$$\left[\int_{S'} U^*(\underline{S'}) \frac{\partial G_1^*(\underline{P}_1, \underline{S'})}{\partial \eta'} dS'\right]\left[\int_{S'} U(\underline{S'}) \frac{\partial G_2^*(\underline{P}_2, \underline{S'})}{\partial \eta'}\right]$$

$$= \left[\int_{S'} U(\underline{S'}) \frac{\partial G_1(\underline{P}_2, \underline{S'})}{\partial \eta'} dS'\right]\left[\int_{S'} U^*(\underline{S'}) \frac{\partial G_2(\underline{P}_1, \underline{S'})}{\partial \eta'} dS'\right]. \quad (3.22)$$

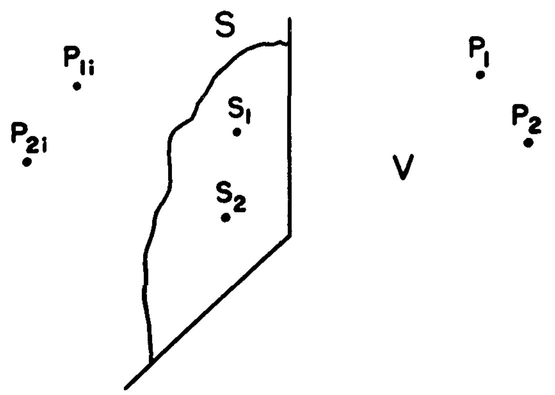

FIG. 1. Geometry For Green's Function.

Since each of the bracketed quantities is a function of one point only, the corresponding quantities can be equated, yielding

$$\int_{S'} U^*(\underline{S}') \frac{\partial G_1^*}{\partial \eta'} (\underline{P}_1, \underline{S}')\, dS' = \int_{S'} U^*(\underline{S}') \frac{\partial G_2}{\partial \eta'} (\underline{P}_1, \underline{S}')\, dS' \tag{3.23}$$

and the desired relationship follows immediately.

4. Statistically Homogeneous Medium

An important case is that in which the refractive index is a stationary (spatially homogeneous) isotropic random process. Assume that the fluctuations, $\epsilon(\underline{P})$, satisfy the relation

$$\overline{\epsilon(\underline{P}_1)\ \epsilon(\underline{P}_2)} = \overline{\epsilon^2(\underline{P})}\ C(\rho).$$

Here $\overline{\epsilon(\underline{P}_1)\ \epsilon(\underline{P}_2)}$ denotes the average of $\epsilon(\underline{P}_1)\,\epsilon(\underline{P}_2)$ taken over all pairs of points $\underline{P}_1$ and $\underline{P}_2$ a fixed distance ρ apart; $\overline{\epsilon^2(\underline{P})}$ is the mean square deviation (of the dielectric constant); and $C(\rho)$ is a correlation function that depends on the separation distance only.

In practice, the most that can be expected from this formulation of the propagation problem is a prediction of effects 'on the average.' For example, suppose there is a plane quasi-monochromatic light source on a slab of ground glass and it is desired to predict the distribution of the mutual coherence function on the far side of the slab. The solution will yield a prediction of the coherence function averaged over a large number of different slabs of glass with the same statistical properties. In other words, the ensemble average distribution of the mutual coherence function can be predicted, but the distribution for a particular slab cannot.

Suppose, then, that a series of measurements are made to determine the average value of $\Gamma(\underline{P}_1, \underline{P}_2, \tau)$ which corresponds to a series of independent but statistically identical samples of a medium with refractive index fluctuations. The source distribution $\Gamma(\underline{S}_1, \underline{S}_2, \tau)$ and the geometrical relations are assumed to be identical for the entire series of measurements. The average can be idealized by letting the number of experiments N become very large and determining the limit of the average as $N \to \infty$. The jth measurement is denoted by $\Gamma_j(\underline{P}_1, \underline{P}_2, \tau)$ and the ensemble average of

$\Gamma(\underline{P}_1, \underline{P}_2, \tau)$, $\left[\Gamma(\underline{P}_1, \underline{P}_2, \tau)\right]$ is defined by

$$\left[\Gamma(\underline{P}_1, \underline{P}_2, \tau)\right] = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \Gamma_j(\underline{P}_1, \underline{P}_2, \tau).$$

With reference to Eq. (3.14), recalling that only ϵ varies from one measurement to another, and with the quasi-monochromatic approximation, the following equation can be written:

$$[\Gamma(\underline{P}_1, \underline{P}_2, \tau)] \simeq e^{-2\pi i \bar{\nu}\tau} \Big\{ \int_{S_2'} \int_{S_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0) \frac{\partial g_1}{\partial \eta_1'} \frac{\partial g_2}{\partial \eta_2'} dS_1' \, dS_2'$$

$$+ k^2 \int_{S_2'} \int_{S_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0) \frac{\partial g_2}{\partial \eta_2'} \int_{V_1''} [\epsilon(\underline{P}_1'')] \, g_1(\underline{P}_1'', \underline{P}_1)$$

$$\times \frac{\partial g_1(\underline{P}_1'', \underline{S}_1')}{\partial \eta_1'} dV_1'' \, dS_1' \, dS_2'$$

$$+ k^2 \int_{S_2'} \int_{S_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0) \frac{\partial g_1}{\partial \eta_1'} \int_{V_2''} [\epsilon(\underline{P}_2'')] \, g_2(\underline{P}_2'', \underline{P}_2)$$

$$\times \frac{\partial g_2(\underline{P}_2'', \underline{S}_2')}{\partial \eta_2'} dV_2'' \, dS_1' \, dS_2'$$

$$+ \ldots$$

where the ensemble average has been taken inside the integrals. Since the samples of the medium associated with the series of experiments are assumed to be independent and statistically homogeneous, the ensemble average of $\epsilon(\underline{P}_1'')$, $\epsilon(\underline{P}_1'')\,\epsilon(\underline{P}_2'')$, and so on (formed with fixed points and different samples of the medium) can be equated with the averages of these same quantities obtained with a particular sample and varying the points (distance relationships are preserved when these enter into consideration). Hence,

$$[\epsilon(\underline{P}_m'')] = \overline{\epsilon(\underline{P})} = 0 \;, \; (m = 1,\ 2)\,,$$

$$[\epsilon(\underline{P}_m'')\epsilon(\underline{P}_m''')] = \overline{\epsilon(\underline{P}_m'')\epsilon(\underline{P}_m''')} = \overline{\epsilon^2(P)}\, C(\rho)\;, \; (m = 1,\ 2)\;, \text{ and so on.}$$

Thus, to second order terms the result that the ensemble average of the mutual coherence function (in the quasi-monochromatic approximation) is given by

$$[\Gamma(\underline{P}_1, \underline{P}_2, \tau)] \simeq e^{-2\pi i\bar{\nu}\tau}\Bigg\{\int_{S_1'}\int_{S_2'} \Gamma(\underline{S}_1', \underline{S}_2', 0)\, \frac{\partial g_1}{\partial \eta_1'}\, \frac{\partial g_2}{\partial \eta_2'}\, d\underline{S}_1'\, d\underline{S}_2'$$

$$+ k^4\, \overline{\epsilon^2(P)} \int_{S_2'}\int_{S_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0)\, \frac{\partial g_2\,(P_2, S_2')}{\partial \eta_2'}$$

$$\times \int_{V_1''}\int_{V_1'''} C(|\underline{P}_1'''-\underline{P}_1''|)\, g_1(\underline{P}_1'''\underline{P}_1'')\, g_1\,(\underline{P}_1'', \underline{P}_1)\, \frac{\partial g_1\,(\underline{P}_1'', \underline{S}_1')}{\partial \eta_1'}$$

$$\times\, dV_1'''\, dV_1'',\, d\underline{S}_1'\, d\underline{S}_2'$$

$$+ k^4\, \overline{\epsilon^2(P)} \int_{S_2'}\int_{S_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0) \int_{V_1''}\int_{V_2''} C(|\underline{P}_1''-\underline{P}_2''|)\, g_1(\underline{P}_1'', \underline{P}_1)$$

$$\times\, g_2(\underline{P}_2'', \underline{P}_2)\, \frac{\partial g_1\,(\underline{P}_1'', \underline{S}_1')}{\partial \eta_1'}\, \frac{\partial g_2(\underline{P}_2'', \underline{S}_2')}{\partial \eta_2'}\, dV_1''\, dV_2''\, d\underline{S}_1'\, d\underline{S}_2'$$

$$+ k^4\, \overline{\epsilon^2(\underline{P})} \int_{S_2'}\int_{S_1'} \Gamma(\underline{S}_1', \underline{S}_2', 0)\, \frac{\partial g_1(\underline{P}_1, \underline{S}_1')}{\partial \eta_1'} \int_{V_2''}\int_{V_2'''} C(|\underline{P}_2'''-\underline{P}_2''|)\, \times$$

$$\times\, g_2(\underline{P}_2''', \underline{P}_2'')\, g_2(\underline{P}_2'', \underline{P}_2)\, \frac{\partial g_2\,(\underline{P}_2''\underline{S}_2')}{\partial \eta_2'}\, dV_2'''\, dV_2''\, d\underline{S}_1'\, d\underline{S}_2'.$$

The statistics of the medium enter into this expression in the mean square of the refractive index fluctuation, $\overline{\epsilon^2(P)}$, and the two-point correlation function $C(\rho)$ which must be integrated over the volume of the medium.

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